A stochastic derivation of the geodesic rule

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Abstract

We argue that the geodesic rule, for global defects, is a consequence of the randomness of the values of the Goldstone field ϕ in each causally connected volume. As these volumes collide and coalescence, ϕ evolves by performing a random walk on the vacuum manifold $\mathcal{M}$. We derive a Fokker-Planck equation that describes the continuum limit of this process. Its fundamental solution is the heat kernel on $\mathcal{M}$, whose leading asymptotic behavior establishes the geodesic rule.

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I. INTRODUCTION

To predict the density of topological defects formed after a phase transition, Kibble [1] postulated two "rules": the randomness and independence of the value of the Goldstone scalars ϕ, parametrize the vacuum manifold $\mathcal{M}$ within each causally connected region, and the "geodesic rule". According to the latter, when two regions of $\mathcal{M}$ having different values of ϕ collide with each other, ϕ interpolates between these values across the common boundary, following the shortest path [1]. The geodesic rule is simple, and as such, aesthetically appealing and favorable for numerical implementation, but it lacks a thorough justification (see however [2],[3]). Moreover, there is no consensus on its validity, or even on its extensions in the case of local defect formation [4]-[8].

We argue that the geodesic rule is a result of the stochasticity of the selection of ϕ within each causal horizon. Our argument relies on the random nature of the collisions between causally unrelated regions on $\mathcal{M}$. It differs from the other treatments on this issue, which rely on energetic considerations [2],[3]. The present discussion applies, as is, to theories with broken global symmetries only. The presence of gauge symmetries would introduce several subtleties in the present arguments. These difficulties are directly related to the fundamental lack of understanding of the vacuum structure of gauge theories. Explicit understanding of the topology and geometry as well as parametrizations of the underlying moduli space of the affine connections of gauge theories is still lacking in most cases of physical interest. As a result, even the formulation of appropriate stochastic equations, analogous to the Fokker-Planck, becomes problematic. For these reasons we will deal only with the case of global defects in this treatment.

The organization of this paper is as follows: Section II presents the derivation of the geodesic rule and the assumptions on which this derivation relies. Section III analyzes the domain of applicability of this result and re-examines its assumptions. Section IV presents a brief discussion and our conclusions.

II. DERIVATION OF THE GEODESIC RULE

At the outset of our treatment, it is worth pointing out that the Goldstone field (order

parameter) ϕ is scalar only as far as its space-time properties are concerned. In a general classical field theory, ϕ is a section of a vector bundle $\mathcal{P}$ over a space-time M which is most frequently considered to be $\mathbb{R}^n$. The typical fiber of $\mathcal{P}$ is a linear representation, in most cases, of a Lie group G [9]. Then ϕ provides a local parametrization of G in $\mathcal{P}$. In a quantum or thermal field theory we start with an appropriate space of maps $f : M \to G$, or, in this language, with an appropriate space of sections of $\mathcal{P}$. These sections must have some physically acceptable regularity properties, most frequently pertaining to their differentiability, as well as appropriate normalization properties with respect to the symmetric or hermitian inner product of G and the Riemannian inner product of M [9]. Moreover, in order to ensure the finiteness of the energy of the system, we impose some asymptotic requirements on f which amount, very frequently, to effectively considering a compactification of M. Such maps tend to form, very frequently, Sobolev spaces on the sections of $\mathcal{P}$.

A classical/non-thermal field theory can be determined by its Lagrangian $\mathcal{L}$ [9],[10]. A quantum/thermal field theory can be defined by a path integral on appropriately defined functional spaces, as was mentioned above. The result of the calculation of such a path integral can be encoded, under some rather general conditions, in such a way as to give rise to extra terms to the classical potential V_c. The requirement of renormalizability [9],[10], when applicable, often imposes additional constraints on the nature of these extra terms, thus severely constraining the form of the initially allowed form of $\mathcal{L}$. Our arguments rely on the existence of such an effective potential V_{eff} [10] for the description of the thermal field theory. Renormalizability constraints of the underlying theory are of no concern here, since we are dealing with effective, as opposed to fundamental, theories from the outset. Then, the effective potential allows us to describe the system in an apparently non-thermal manner and that is why many of our arguments have a very strong classical/non-thermal flavor.

For concreteness, we begin by considering a first order phase transition proceeding by bubble nucleation [11]. This line of argument essentially holds also for continuous transitions proceeding by spinodal decomposition, as we comment at the end of the paper. We assume that the phase transition giving rise to the topological defects proceeds by thermalization and will be eventually completed. It is known [12],[13] that after their formation, bubbles whose radius is larger than a critical value r_{crit} tend to expand and

bubbles whose radius is smaller than r_{crit} tend to shrink until they disappear. For this reason, we restrict our attention to bubbles of radii greater than r_{crit}. We are interested in length scales much bigger than r_{crit}. Then without loss of generality we can pretend that r can take any positive value with lower bound zero.

A first assumption is that when two bubbles collide and coalesce, the bubble which is generated has an effective value of ϕ which is the weighted average of the values of the two bubbles that collided. In this weighted average, the contribution of each of the two colliding bubbles should be proportional to the number of points it contains, i.e. to its volume. It is simplicity that makes such a choice plausible. It is very possible that there are systems for which such a rule for calculating the weighted average does not apply, although we fail to see a generic reason why it should not. Thus, if the two coalescing bubbles have values of Goldstone fields ϕ_1 and ϕ_2 respectively, the resulting bubble will have an effective value of ϕ given by

$$\phi = \frac{V_1\phi_1 + V_2\phi_2}{V_1 + V_2} \tag{1}$$

where V_1 and V_2 are the corresponding volumes of each bubble.

A second assumption is that at least two distinct, widely separated time scales exist for such a system: τ_C which describes the average time between two consecutive bubble collisions and τ_M which is the mixing time of the two values of the Goldstone fields, after the bubble collision has taken place. These time scales are determined by the dynamics of each model. Their specific values, important as they are in the dynamics of phase transitions, are not relevant in our argument, so we will not delve into the issue of their calculation any further. This assumption is very strong as we expect that most systems violate it, more or less. However, as we explain in section III, for systems that follow it or weakly violate it, the geodesic rule should still hold, due to the wide domain of applicability of the Central Limit theorem. If $\tau_C \ll \tau_M$, the values of ϕ of the two bubbles will not have enough time to mix well between two collisions. This is a case favorable for the formation of topological defects [14],[15]. The opposite case, namely $\tau_C \gg \tau_M$, is very unfavorable for the formation of defects. For this reason we examine systems that belong to the former case. Incidentally, the probability of simultaneous collisions of three or more bubbles is so small that the contribution of such events to the number of produced defects

is negligible.

Due to the reasons given in the first two paragraphs of this section, we need to describe aspects of the dynamics of the phase transition as it evolves through bubble collisions on $\mathcal{M} \times \mathbb{R}$. This evolution is parametrized by a scalar variable $t \in \mathbb{R}$. The relation between t and the actual time in which the phase transition takes place depends on the parameters of the specific model. We do not need to know any further details about such a relation for our argument's sake; it just suffices to know that such a relation exists. Because of the random distribution of bubbles and the random way in which they collide, ϕ undergoes a random evolution i.e. it undergoes a random walk on $\mathcal{M}$. Let $P(\phi, t)$ denote the probability density function describing this stochastic process. To determine $P(\phi, t)$, we start by discretizing $\mathbb{R}$, i.e. reducing it to a Euclidean one dimensional lattice, isomorphic to $\mathbb{Z}$. We set the spacing of this lattice to be equal to τ_C since this is the shortest scale of t associated with the evolution of bubbles. Since $\tau_M \gg \tau_C$, we can assume without loss of generality that τ_M is an integral multiple of τ_C. Then $P(\phi, t)$ has domain $\mathcal{M} \times \mathbb{Z}$. To explicitly indicate this discretization, we rewrite t is as t_n, where $t_n \in \mathbb{Z}$.

A third assumption in our argument is that the evolution of the system of bubbles described by $P(\phi, t_n)$ is a Markovian process. This is actually an oversimplification. We have already assumed that $\tau_M \gg \tau_C$, so after each collision the values of ϕ do not mix very strongly in the resulting bubble. Then the relative orientation of the bubbles before the collision took place leaves its trace on the resulting bubble through the distribution of the values of ϕ on it. However, even within our approximation, the relative orientation of two bubbles before each collision is random. Eventually, we average over all possible collisions between pairs of bubbles. The averaging process erases all trace of the relative orientation of bubbles before a collision. In short, although the collision between two specific bubbles would give rise to a non-Markovian process, the randomness of orientation and the averaging process reduce the stochastic process to a Markovian one. This stochastic process is determined by $P(\phi, t_n)$ which obeys the Chapman-Kolmogorov equation [16],[20]

$$P(\phi_3, t_3) = \int_{\mathcal{M}} P(\phi_3, t_3 | \phi_2, t_2)\ P(\phi_2, t_2 | \phi_1, t_1)\ d\phi_2 \tag{2}$$

where $P(\phi_2, t_2 | \phi_1, t_1)$ denotes the conditional probability density for the stochastic variable to have value ϕ_2 at t_2 given that it had value ϕ_1 at t_1. The definition of the probability

density function $P(\phi, t_n)$ also gives

$$P(\phi, t_n + \tau_C) = \int_{\mathcal{M}} P(\phi, t_n + \tau_C | \psi, t_n)\ P(\psi, t_n)\ d\psi \tag{3}$$

We are interested in the behavior of the system at large t scales when compared to τ_C, as the system approaches its equilibrium, so we have to consider the continuum time limit $\tau_C \to 0$. Then t_n is replaced by t and a Taylor series expansion with respect to τ_C gives the master equation [16]

$$\frac{\partial P(\phi, t)}{\partial t} = \int_{\mathcal{M}} \{W(\phi|\psi)P(\psi, t) - W(\psi|\phi)P(\phi, t)\}\ d\psi \tag{4}$$

Here $W(\psi|\phi)$ denotes the transition probability per unit t for the Goldstone field to attain the final value ψ, given its initial value was ϕ.

A considerable simplification occurs, if we further assume that each bubble collides only with much smaller ones. This is a very strong assumption, which we expect to be violated in may cases. Clearly the way the bubbles evolve and collide depends on the details of the nucleation model describing the bubbles. It is moreover true that generally many bubbles of similar sizes collide with each other, a fact which renders this approximation invalid. The assumption is necessary though, in order to make the quantitative analysis tractable. We expect that because of the very wide range of applicability of the Central Limit theorem, the final conclusion that we reach applies to many cases that initially may even violate this assumption. We perform a Taylor series expansion in terms of the short displacement s that the scalar field ϕ experiences after each collision. Complications arise because $\mathcal{M}$ is a Riemannian manifold which is not flat. The equations come as close as possible to the ones of the flat case by choosing a normal coordinate system. In this coordinate system the connection coefficients (Christoffel symbols) are zero at the origin O and, to a first order approximation the metric there is Euclidean. Let $\{e_1, e_2, \ldots, e_{\dim \mathcal{M}}\}$ be a coordinate basis of $T_o\mathcal{M}$. The master equation (4), when expanded up to quadratic terms in s in this basis [20],[21] gives

$$\frac{\partial P(\phi, t)}{\partial t} = -\nabla_i \left[a^i(\phi)P(\phi, t)\right] + \frac{1}{2}\nabla_i\nabla_j \left[a^{ij}(\phi)P(\phi, t)\right] \tag{5}$$

where

$$a^{ij}(\phi) = \int_0^{+\infty} s^i s^j\ W(\phi; s)\ ds \tag{6}$$

with

$$W(\psi|\phi) = W(\phi; s) \tag{7}$$

where s is the distance between ϕ and ψ with respect to the metric of $\mathcal{M}$, and ∇ is a Riemannian connection on $\mathcal{M}$, not necessarily compatible with the metric [19]. This Fokker-Planck equation (5) has the first moment $a^i(\phi)$ equal to zero since there are no "external fields", i.e. preferential directions on $\mathcal{M}$. Even if that is not the case, and the linear term is non-zero, the stochasticity of the directions of the bubble collisions would eliminate such a linear term upon averaging over $T_o\mathcal{M}$. Then (5) reduces to the diffusion equation

$$\frac{\partial P(\phi,t)}{\partial t} = \nabla_i \nabla_j \left[D^{ij}(\phi) P(\phi,t) \right] \tag{8}$$

with $D^{ij}(\phi) = \frac{1}{2}\, a^{ij}(\phi)$

If a bubble under consideration collides with either similar in size or much bigger bubbles, then the approximation leading to (5) is insufficient to describe the phenomenon. In such a case we need to keep more terms in the Kramers-Moyal expansion

$$\frac{\partial P(\phi,t)}{\partial t} = \sum_{n_{i_1} n_{i_2} \cdots n_{i_k} \cdots}^{\infty} \frac{(-1)^n}{n_{i_1}! n_{i_2}! \cdots n_{i_k}! \cdots} \nabla_{i_1} \nabla_{i_2} \cdots \nabla_{i_k} \cdots \left[a^{i_1 i_2 \cdots i_k \cdots}(\phi) P(\phi,t) \right] \tag{9}$$

of the master equation (4). Actually, according to Pawula's theorem [20], which holds for a stochastic process in Euclidean space, the positivity of the transition probability rate, $W(\psi|\phi)$, implies that the Kramers-Moyal expansion either terminates at the second order term (resulting in the Fokker-Planck equation (5)) or never terminates. Considering this theorem, our approximation leading to the termination of the Kramers-Moyal expansion (9) at the second order term is also optimal, in a sense, for technical reasons: any other choice would result in infinite sub-harmonic terms which would not be manageable without further, model-dependent, approximations. It is worth mentioning at this point that the expansion (9) is used heuristically. The covariant derivatives on $\mathcal{M}$ do not commute [19], since

$$\nabla_i \nabla_j \; e_k - \nabla_j \nabla_i \; e_k = R(e_i, e_j) \; e_k \tag{10}$$

where $R(e_i, e_j)e_k$ are the Riemann tensor components in the basis of $T_o\mathcal{M}$ that we are using. The ambiguity in the order of the covariant derivatives appearing in the terms of order higher than two in (9), result in extra terms containing the Riemann tensor and

its covariant derivatives. The second order term may, at worst, acquire an extra term involving the scalar curvature of $\mathcal{M}$, a fact that does not alter the leading dependence of the resulting solution (13) on the distance function $d(\phi)$, as will be seen in the sequel. The order, as well as the number of times, that the "diffusion" tensor components $a^{i_1 i_2 \cdots i_k \cdots}(\phi)$ have to be differentiated depend on the details of the system. These ambiguities should be considered as an addition to the, essentially arbitrary at the mesoscopic level, choice of the Itô or the Stratonovich way of performing the stochastic integration [16],[21], which arises in the integration of the Fokker-Planck equation (5). In either case, a more detailed knowledge of the microscopic behavior of the system is required [16],[20] for the correct choices to be made. Such a particular choice, however, plays no role in our subsequent arguments.

In many cases of physical interest, the system can be described by averaged diffusion coefficients $D^{i_1 i_2 \cdots i_n \cdots}$. Once more, the averaging process over $T\mathcal{M}$ for bubble collisions justifies this fact. Then, we can replace the different values of the tensor components by a family of scalar functions $\{D_l(\phi), l = 1, 2, \ldots\}$ in each term of (9). A class of examples of vacua very frequently encountered in field theoretical applications for which such a reduction is possible is provided by homogeneous spaces [19] of compact Lie groups. Moreover, we make the additional assumption that the function $D(\phi)$, to which the coefficient of (8) reduces, varies slowly over $\mathcal{M}$. This allows us to ignore the contribution of the derivatives of D and (8) reduces to the diffusion equation

$$\frac{\partial P(\phi, t)}{\partial t} = D \nabla^2 P(\phi, t) \tag{11}$$

where ∇^2 denotes the Laplace-Beltrami operator associated with the metric of $\mathcal{M}$. Examples of a slow variation of $D(\phi)$ are $\mathcal{M}$ which are symmetric spaces [19]. Even more specifically, when $\mathcal{M}$ is a compact quotient of a sphere or a torus by the free action of a discrete group, as in the case of projective spaces for instance, then $D(\phi)$ is exactly constant [22].

The solution of (11) with the initial condition

$$P(\phi, 0) = \delta(\phi) \tag{12}$$

is the heat kernel of the Laplace-Beltrami operator ∇^2 on $\mathcal{M}$. Using the maximum principle one can prove that $P(\phi, t)$, which is really the heat kernel of ∇^2 on $\mathcal{M}$, is

positive definite [21],[23]. Then its normalized form can be legitimately called a probability density function, as we have been assuming all along. We are interested in the behavior of the system for relatively short times t. Indeed, at very long t the effects of mixing will have become significant and, as was remarked after (1), this would introduce additional complications that could mask or even invalidate this line of argument. At small t, i.e. as $t \to 0^+$ the heat kernel has the asymptotic expansion [21],[23]

$$P(\phi) = (4\pi t)^{-\frac{\dim \mathcal{M}}{2}} \exp\left(-\frac{[d(\phi)]^2}{4t}\right) \eta(d(\phi)) \left\{ \sum_{i=0}^{k} u_i(0) t^i + o(t^k) \right\} \tag{13}$$

where $\eta(d(\phi))$ is a bump function which has the value one, if ϕ is in the cut-locus of the origin O, and zero if ϕ is outside it. We can then clearly see [21] that the leading behavior of $P(\phi)$ as $t \to 0^+$, is

$$\lim_{t \to 0^+} t \ln P(\phi) = -\frac{[d(\phi)]^2}{2} \tag{14}$$

which is actually true for any $\phi \in \mathcal{M}$ (Varadhan's asymptotic formula) [21].

Let the initial and final values of ϕ on $\mathcal{M}$ be fixed. Then, according to (14), the stochastic process described by (2) and (3) gives rise to a probability density $P(\phi, t)$ which is maximized if the distance function $d(\phi)$ between O and ϕ is minimized, i.e. along the minimal geodesic joining them. This observation is the geodesic rule.

III. ON THE APPLICABILITY OF THE GEODESIC RULE

Let l_{min} and l be the lengths of the minimal geodesic and some other piecewise smooth curve of $\mathcal{M}$ joining O and ϕ, respectively, and let $P_{min}(\phi)$ and $P(\phi)$ the corresponding probability densities at ϕ in the limit $t \to 0^+$. Then (13) implies that

$$P(\phi) = P_{min}(\phi) \exp\left\{-\frac{l^2 - l_{min}^2}{4t}\right\} \tag{15}$$

so the contribution of non-minimal paths is exponentially suppressed in expressions like path integrals. It is worth noticing that the geodesic rule can actually be violated. We have argued that it is not really a rule, but rather a minimalist prescription for calculating the density of global topological defects. If someone is interested in calculating the number (density) of topological defects with some degree of accuracy beyond a simple

estimate, either analytically or numerically, the geodesic rule will provide a fast and accurate way most of the times. However, we expect deviations from its predictions to become important in cases in which there are many paths for which $P(\phi) \approx P_{min}$. One such case occurs, if, for instance $\mathcal{M}$ is a C_l manifold [19], i.e. a compact Riemannian manifold all the closed geodesics of which have length l. Examples of C_l manifolds are symmetric spaces of positive sectional curvature, manifolds which occur frequently as vacua of physical systems.

What we have effectively done in this approach, is that we have arrived at an integration measure on the appropriate space of functions on $\mathcal{M}$, namely the Wiener measure on this space [24]. This is then used to perform the thermal path integral describing the phase transition which gives rise to the topological defects. The path integral is essentially determined by the Feynman-Kac formula [24], [25]

$$ker\ e^{-t\mathcal{H}} = \int e^{-tV(t)dt} d\phi \tag{16}$$

for the kernel of $e^{-t\mathcal{H}}$, where $\mathcal{H}$ denotes the effective Hamiltonian on $\mathcal{M}$ of the system of colliding bubbles. Then, in the normal coordinate system that we are using and disregarding any curvature terms, which provide second order corrections in t and are eventually factorized in a path integral, $V(t)$ is trivial on the path space of $\mathcal{M}$. Therefore, the dominant contribution in the thermal path integral comes from evaluation of the Laplacian on the shortest possible paths on $\mathcal{M}$, a fact which establishes the geodesic rule.

The result of this analysis relies on establishing, for this system, the probability distribution given by (13). This probability distribution is Gaussian with respect to the distance function $d(\phi)$. The Central Limit theorem [26] states that under some very general conditions a probability density on $\mathcal{M}$ would converge to (13). We are then led to believe that although (13) is derived under some rather strong assumptions, it should be valid even in cases that violate one or more of these assumptions, as long as the assumptions of the Central Limit theorem are obeyed. The geodesic rule describes the leading, universal, behavior of such systems. One class of systems that violate the geodesic rule are ones whose bubble collisions are described by a non-Markovian process such as, for instance, a Levy process [17]. Even in such cases, variations of the Generalized Central Limit theorem still hold, a fact however which does not guarantee the validity of the

geodesic rule in its original formulation, which is the one that we have proved in this paper.

A case in which it is known that the geodesic rule does not hold is when the dynamics of the order parameter is dominated by fluctuations [4],[8],[27]. Such an observation however does not contradict our conclusions. Indeed, when fluctuations dominate, the absolute minimum of the effective potential [10] is what determines $\mathcal{M}$. The loop expansion of the effective potential, in powers of $\hbar$ or a corresponding thermal parameter [10], shows that fluctuations force the effective potential to develop singularities [10]. The nature of these singularities is not clear, but it becomes amply evident through such a procedure that $\mathcal{M}$ can no longer be considered to be a Lie group, a homogeneous space or even a manifold as in the classical/non-thermal case [22]. As a result, all the arguments leading to (11) and subsequently to (13) and (15), which rely on $\mathcal{M}$ being a Riemannian manifold do not hold any more. We conclude then that the treatment of cases of dominant fluctuations lies outside the scope of our approach, hence there is no contradiction with our results.

IV. DISCUSSION AND CONCLUSIONS

In the case of a first-order phase transition [11] there is an energy barrier that ϕ has to overcome in order to move from the meta-stable to the stable vacuum. If we consider the Fourier transform of the Fokker-Planck equation, then we expect that $\tilde{D}(k)$ will have a strong peak on wave-numbers k comparable to the width of the potential well around the meta-stable vacuum. In the case of continuous transitions (spinodal decomposition), there is no longer an energy barrier that would act as a localizing factor and would force $\tilde{D}(k)$ to have sharp peaks. The arguments presented above do not use this localization in momentum space, therefore they are equally valid in both cases. This, of course, does not mean that the two cases are physically identical. In a continuous transition, we can still consider a region that shares the same value of the scalar field, but does not possess a physical boundary, like a bubble wall, as in the case of first order phase transitions.

To conclude, we have shown that the geodesic rule holds for a general class of systems undergoing a phase transition, as a leading approximation. Violations of the rule can and will generally occur. The validity of this rule depends on establishing a Gaussian process through which the phase transition from the meta-stable to the stable vacuum

proceeds. For any system that can be described by such a stochastic process, which obeys the assumptions of the Central Limit theorem, the density of the resulting global defects can be accurately estimated by using the geodesic rule.

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